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Note

On the random generation and counting of matchings in dense graphs [1]

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Abstract

In this work we present a fully randomized approximation scheme for counting the number of perfect matchings in a dense bipartite graphs, that is equivalent to get a fully randomized approximation scheme to the permanent of a dense boolean matrix. We achieve this known solution, through novel extensions in the theory of suitable non-reversible, Markov chains which mix rapidly and have a near-uniform distribution. © 1998—Elsevier Science B.V. All rights reserved

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1. Introduction

Many problems involving the counting of the number of solutions of combinatorial structures, are well known to be in the difficult class $\#P$ and their exact counting is apparently intractable [10]. The most notorious of these problems is to compute the permanent of a dense matrix, that turns out to be equivalent to count the number of perfect matchings in a dense bipartite graph. The hardness of these counting problems has led to approximate the counting. Pioneering work in this line was the paper [5] where they construct a Randomized Fully Approximation Scheme for some difficult counting problems. Later, it was discovered that for the problems which have certain structural property, approximate counting is equivalent to almost uniform generation [4]. The almost uniform generation problem consist in picking at random an element of a

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finite set characterized by some property, with a relative error of at most ε with respect to the probability that a given element is chosen.

A technique that has proved to be very useful to solve the almost uniform generation problem, is the Markov Chain technique. Given a problem, define a Markov chain where the states are all possible solutions, plus possibly a small fraction of "near-solutions", and the transitions are certain probabilistics rules that allow us to remain in the same state or to pass to a new state. Under certain properties of the underlying graph, it can be proved that a polynomial (in the input size) random walk on the states give us an almost randomly generated element from the stationary distribution of the Chain. The difficulty of this method is to prove rapid convergence to the stationary distribution, what is called the "rapid mixing" property. Broder used the Markov chain technique to approximate the value of the permanent of a dense matrix [1]. The rapid mixing property of his chains was shown by Jerrum and Sinclair [3], and their proof relies heavily on the reversibility of the Markov chain [11].

In this note, we show how to use non-reversible Markov chain to obtain the same result of Jerrum–Sinclair.

2. Basic definitions

Let us introduce some basic definitions and results on Markov chains, for further details see [11].

Definition 1. A Markov chain $\mathscr{M}$ is an stochastic process, defined on a set of states S, in terms of a transition matrix $P=(p_{ij})_{i,j\in S}$, where each p_{ij} denotes the probability of going from i to j. So,

$$\forall i\in S,\quad \sum_{j\in S} p_{ij}=1.$$

Moreover, at $t=k$, we define π_k as

$$\pi_k(i)=\sum_{j\in S} p_{ji}\cdot\pi_{k-1}(j). \tag{1}$$

Definition 2. A Markov chain is *irreducible* if $\forall i,j\in S$, $\exists t$ such that $p_{ij}^t>0$. That is, from any state we can go to any other state with positive probability after enough time. A Markov chain is aperiodic if $\forall i,j\in S$, $\gcd\{t \mid p_{ij}^t>0\}=1$. A Markov chain is said to be *ergodic* if $\forall j\in S$, $\lim_{t\to\infty} p_{ij}^t=\pi_\infty(j)>0$.

So, if the chain is *ergodic* then $\overline{\pi}_\infty=(\pi_\infty(1),\ldots,\pi_\infty(n))$ is called the *limit distribution* or *stationary distribution*.

Theorem 1. *If a Markov chain $\mathscr{M}$ is ergodic then $\overline{\pi}_\infty$ (i.e. the stationary distribution) is the unique distribution that satisfies*

$$\overline{\pi}_\infty\cdot P=\overline{\pi}_\infty,$$

$$\sum_{i\in S}\pi_\infty(i)=1.$$

We will usually refer to the first condition, in the previous theorem by the term *the balance equation.*

Theorem 2. *A Markov chain μ is ergodic if and only if it is irreducible and aperiodic.*

Definition 3. A Markov chain is *symmetric* if $\forall i,j\in S, p_{ij}=p_{ji}$ (P is doubly stochastic).

An ergodic Markov chain is *reversible* if $\forall i,j\in S$ satisfy the Balance Equation

$$\pi_\infty(i)p_{ij}=\pi_\infty(j)p_{ji}.$$

Given a graph $G=(V,E)$, a *matching* is a subset of edges such that no two edges have a vertex in common. The size of a matching is the number of edges in it. Let $[n]$ denote the set $\{0,\dots,n-1\}$. For $k\in[n]$, $M_k(G)$ will denote the set of matchings of size k in G. From now on, G will denote the input graph. It is well known that testing if G has a perfect matching is in RNC [9].

We will consider bipartite graphs of the form $G=(V_1,V_2,E)$, $|V_1|=|V_2|=n$. We say that a bipartite graph is dense if every vertex is connected to at least $n/2$ vertices.

3. A near-uniform, non-reversible Markov chain for M_n and M_{n-1}

Let $G=(V_1,V_2,E)$ be a dense bipartite graph, with $|V_1|=V_2|=n$. From now on, G will denote the input graph. Let $\mathscr{M}_1$ be the Markov chain defined by Broder [7] for the almost uniform generation of elements in $M_n\cup M_{n-1}$.

Definition of Markov chain $\mathscr{M}_1$ on state-space $M_n\cup M_{n-1}$: Given a matching $m\in M_n\cup M_{n-1}$ and a uniformly sampled random edge $e=(u,v)$.

(1) With probability $\frac{1}{2}$ remain in the same state, otherwise

(2) If $m\in M_n$ and $e\in m$ then go to near perfect matching $m-e$, otherwise stay in m.

(3) If $m\in M_{n-1}$,

(3.1) if u and v are unmatched in m, move to perfect matching $m\cup e$,

(3.2) if u is matched to w and v is unmatched, delete (u,w) from m and add e, to arrive to a new state,

(4) Otherwise, do nothing.

We modify $\mathscr{M}_1$ to obtain a second Markov chain $\mathscr{M}_2$ in the following way: Use the same set of states $M_n\cup M_{n-1}$ and modify the transitions,

Definition of Markov chain $\mathscr{M}_2$ on state-space $M_n\cup M_{n-1}$: Given a matching $m\in M_n\cup M_{n-1}$ and a uniformly sampled random edge $e=(u,v)$ and let f be a sufficiently small function of n (through the paper we will get tight upper bounds to its value),

(1) With probability $\frac{1}{2}$ remain in the same state, otherwise

(2) If $m\in M_n$

(2.1) If $e\in m$ then go to near perfect matching $m-e$, otherwise

(2.2) If there is a connected component of length 3, this component must be formed by e and two edges in m, e_1 and e_2.
(2.2.1) with probability $1-f$ remove only the edge e. Otherwise, either
(2.2.2) with probability $f/4$ choose to remove e and e_1, or
(2.2.3) with probability $f/4$ choose to remove e and e_2, or
(2.2.4) with probability $f/2$ choose to keep the new e (and remove e_1 and e_2).
(3) If $m \in M_{n-1}$,
(3.1) if u and v are unmatched in m, move to perfect matching $m \cup e$,
(3.2) if u is matched to w and v is unmatched, delete (u,w) from m and add e, to arrive to a new state.
(4) Otherwise, do nothing.

In plain words, $\mathcal{M}_2$ is the $\mathcal{M}_1$ chain except that for states representing perfect matchings we have some extra transitions, called *f-transitions*, with a certain probability depending on f. Notice that we keep in $\mathcal{M}_2$ the same connections in $\mathcal{M}_1$, with at least the same probability, except for the self-loop probability of elements in M_n that may decrease.

Given a finite sample space Ω, an ergodic and aperiodic Markov chain $\mathcal{M}$ with steady state distribution P is said to be *near-uniform* with error c if for every $\omega \in \Omega$,

$$\frac{|P(\omega)-1/|\Omega||}{1|\Omega|} \leqslant c.$$

Notice this definition is equivalent to Definition 11.6 in Motwani and Raghavan [8].

From now, when dealing with equations, let $p^{(i)}$ represent transitions probabilities in the chain $\mathcal{M}_i$, $i \in \{1,2\}$.

Theorem 3. *The Markov chain $\mathcal{M}_2$ is ergodic and aperiodic on finite state space $\Omega = M_n \cup M_{n-1}$. Moreover, $\mathcal{M}_2$ is near-uniform with error $c \leqslant O(kf)$ where k is the maximum number of f-transitions going out of any perfect matching state in the chain.*

Proof. To see that $\mathcal{M}_2$ is ergodic and aperiodic, just take into account that there is some positive probability of going from any state to any other state in n steps (just as in $\mathcal{M}_1$), and that the self-loop probability of each element is at least $\frac{1}{2}$.

To show that the stationary distribution is almost uniform, we use the balance equations.

Let $m_i \in M_n$ be a state in $\mathcal{M}_2$. Consider the balance equation for m_i, $p_i^{(2)} = \sum p_{ji}^{(2)} p_j^{(2)}$ corresponding to going from state m_j to m_i. But in $\mathcal{M}_1$ we also must have $p_i^{(1)} = \sum p_{ji}^{(1)} p_j^{(1)}$, with the difference between probability transitions between the same states in $\mathcal{M}_1$ and $\mathcal{M}_2$ due to the existence of f-transitions in $\mathcal{M}_2$. Moreover, as k is the number of f-transitions going out of m_i, we have $p_{ii}^{(2)} = p_{ii}^{(1)} - k\Theta(f)$. In the case the transition from m_j to m_i does not have an f-transition, we get that $p_{ji}^{(2)} = p_{ji}^{(1)}$.

Therefore letting $\Delta p_i = p_i^{(2)} - p_i^{(1)}$ we get

$$\begin{aligned}\Delta p_i = &\sum_{\text{not f-trans}} p_{ji}^{(1)}\Delta p_j + p_{ii}^{(2)}p_i^{(2)} - p_{ii}^{(1)}p_i^{(1)} \\ &+ \sum_{\text{f-trans}} p_{ji}^{(1)}\Delta p_j \\ = &\sum p_{ji}^{(2)}\Delta p_j + \sum p_{ji}^{(1)}\Delta p_j + p_{ii}^{(1)}\Delta p_i - k\Theta(f)(p_i^{(1)} + \Delta p_i),\end{aligned}$$

therefore we get

$$\Delta p_i(1 - p_{ii}^{(1)} - k\Theta(f)) = \sum_{m_j \in M_n} p_{ji}^{(2)}\Delta p_j + \sum_{m_j \in M_{n-1},\, j \neq i} p_{ji}^{(1)}\Delta p_j - k\Theta(f)p_i^{(1)}. \quad (2)$$

Plugging the known value $p_i^{(1)} = 1/(|M_n| + |M_{n-1}|)$ in the above equation and choosing

$$\Delta p_i = k\Theta(f)\frac{\Theta(1)}{|M_n| + |M_{n-1}|} \quad \text{and} \quad \Delta p_j \leqslant k\Theta(f)\frac{\Theta(1)}{|M_n| + |M_{n-1}|}$$

which satisfies Eq. (2). Manipulating in the same way the balance equation for $m_j \in M_{n-1}$ we get

$$\Delta p_j = \sum p_{ij}^{(1)}\Delta p_i + \sum_{m_i \in M_n} p_i^{(2)}\Theta(f) + p_{jj}^{(1)}\Delta p_j$$

again satisfied by

$$\Delta p_j = k\Theta(f)\frac{\Theta(1)}{|M_n| + |M_{n-1}|}\Delta p_i \leqslant k\Theta(f)\frac{\Theta(1)}{|M_n| + |M_{n-1}|}. \qquad \square$$

Notice that for every state m_i in $\mathcal{M}_2$ we have $p_{ii}^{(2)} \geqslant \frac{1}{2}$, therefore $\mathcal{M}_2$ is strongly aperiodic.

Let us recall Theorem 3.1 of [6].

Theorem 4 (Mihail). *For general non-reversible and strongly aperiodic Markov chains, let $\vec{x}(t)$ denote the probability distribution of the states of the chain at time t, let $\vec{\pi}$ the stationary distribution of the chain, let define the discrepancy at time t as $\vec{e}(t) = \vec{x}(t) - \vec{\pi}$, then*

$$\|\vec{e}(t)\| \leqslant (1 - \Phi^2)^t \|\vec{e}(0)\|,$$

where the conductance $\Phi = \min_{A \subseteq \Omega} \Phi(A)$ with A such that $\sum_{i \in A} \pi_i \leqslant \frac{1}{2}$ and

$$\Phi(A) = \frac{\sum_{i \in A}\sum_{j \in \Omega - A} \pi_i p_{ij}}{\sum_{i \in A} \pi_i}.$$

To show that $\mathcal{M}_2$ mixes rapidly, we just need to prove that the conductance Φ of the chain $\mathcal{M}_2$ is greater than 1 over a polynomial in n, as the strong aperiodicity of the chain allows us to drop the condition of reversibility in the standard use of the conductance.

Theorem 5. *The conductance Φ_2 of the chain $\mathcal{M}_2$ satisfies $\Phi_2 \geqslant 1/\text{poly}(n)$. Thus, $\mathcal{M}_2$ is rapidly mixing, in the sense that at time $t = \text{poly}\ (n)$, $\|\vec{e}(t)\| \leqslant (1/\text{poly}(n))\|\vec{e}(0)\|$.*

Proof. From Theorem 3, we know that for any state m in the chain $\mathcal{M}_2$, there exists constants $c_1, c_2 > 1$ such that

$$\frac{c_2}{|\Omega|} + \frac{1}{|\Omega|} \leqslant \pi(m) \leqslant \frac{1}{|\Omega|} + \frac{c_1}{|\Omega|},$$

where $\Omega = M_n \cup M_{n-1}$. Therefore, for all sets $A \subseteq \Omega$,

$$\Phi(A) \geqslant \frac{|\frac{1}{|\Omega|} - \frac{c_2}{|\Omega|}| \sum_{m_i \in A} \sum_{m_j \in \Omega - A} p_{ij}}{(\frac{1}{|\Omega|} + \frac{c_1}{|\Omega|}) \sum_{m_i \in A} 1}$$

which implies that $\Phi(A) \geqslant (c_2 - 1/c_1 + 1) \sum_{m_i \in A} \sum_{m_j \in \Omega - A} p_{ij}/|A|$.

But notice that we always have $p_{ij} \geqslant \min\{f/c_3, 1/2|m|\}$ where $c_3 > 1$, $f = 1/\text{poly}(n)$, therefore

$$\Phi(A) \geqslant \frac{c_2 - 1}{c_1 + 1} \frac{1}{\text{poly}(n)} \frac{(\text{number of edges out of } A)}{|A|}.$$

We still have to show that the number of edges between A and $\Omega - A$ is at least $|A|/\text{poly}(n)$. But notice that for any state of $\mathcal{M}_2$ there are always at least as many edges out as in $\mathcal{M}_1$, because we have just added some edges, therefore our statement follows from the canonical path argument of Jerrum and Sinclair for $\mathcal{M}_1$ [3]. □

4. A non-reversible Markov chain for all matchings

Let us define a new Markov chain $\mathcal{M}_3$ this chain will have as state-space the set $\Omega = M_n \cup M_{n-1} \cup \cdots \cup M_1$ of all matchings in G. We shall show that the stationary distribution of $\mathcal{M}_3$ is "close", to $\mathcal{M}_2$ as far as perfect or near perfect matchings are concerned, with respect to the steady state probabilities.

Definition of the Chain. $\mathcal{M}_3$ on state-space Ω:

(1) With probability $\frac{1}{2}$ do nothing, else
(2) If $m \in M_n$
  (2.1) If $e \in m$ then go to near perfect matching $m - e$, otherwise
  (2.2) If there is a connected component of length 3, this component must be formed by e and two edges in m, e_1 and e_2.
    (2.2.1) with probability $1 - f$ remove only the edge e. Otherwise, either
    (2.2.2) with probability $f/4$ choose to remove e and e_1, or
    (2.2.3) with probability $f/4$ choose to remove e and e_2, or
    (2.2.4) with probability $f/2$ choose to keep the new e (and remove e_1 and e_2).

(3) If $m \in M_k$, with $1 \leqslant k \leqslant n-1$,

(3.1) if u and v are unmatched in m, move to perfect matching $m \cup e$,

(3.2) if u is matched to w and v is unmatched, delete (u, w) from m and add e, to arrive to a new state,

(3.3) otherwise, there is a connected component of length 3, do as in (2.2) above.

(4) Otherwise, do nothing.

Theorem 6. *The chain $\mathcal{M}_3$ is ergodic, it has a finite state space and it is strongly aperiodic.*

Proof. Notice that for any $m_i \in M_k$, $1 \leqslant k \leqslant n-1$ we have $p_{ii} > \frac{1}{2}$ because by definition it is equal to $\frac{1}{2}$ plus the probability that we have greater than zero of idle moves. Then the statement of the theorem follows from the arguments in the previous section. □

Theorem 7. *The steady-state probabilities of $\mathcal{M}_3$ are as follows:*

(1) *For $m \in \{M_n \cup M_{n-1}\}$,*

$$\left| \Pr\{m\} - \frac{1}{|M_n| + |M_{n-1}|} \right| \leqslant \frac{\mathrm{O}(hf)}{|M_n| + |M_{n-1}|},$$

where h is the number of transitions from a particular $m_j \in M_{n-1}$ to any $m_l \in M_{n-2}$.

(2) *$\exists c_3 > 1$ such that $\forall m \in M_k$, $k \leqslant n-2$, we get $\Pr\{m\} \leqslant c_1 c_3 / (|M_n| + |M_{n-1}|)$.*

Proof. Consider the balance equations for state $m_j \in M_{n-1}$, using the previous notation, the transition probabilities in $\mathcal{M}_3$ will carry a (3), super-index. Then,

$$p_j^{(3)} = \sum_{m_k \in M_n} p_{kj}^{(2)} p_k^{(3)} + p_{jj}^{(3)} p_j^{(3)} + \sum_{m_i \in M_{n-1},\, i \neq j} p_{ij}^{(2)} p_i^{(3)} + \sum_{m_l \in M_{n-2}} p_{lj}^{(3)} p_l^{(3)} \tag{3}$$

but due to f-transitions we have $p_{jj}^{(3)} = p_{jj}^{(2)} + h\Theta(f)$, where h is the number of transitions from $m_j \in M_{n-1}$ to any $m_l \in M_{n-2}$, so $h \leqslant \mathrm{O}(n^2)$.

Note also that $p_{lj}^{(3)} = \Theta(1/m)$ because there is exactly one edge e that can cause this transition to happen.

From $\mathcal{M}_2$ we get

$$p_j^{(2)} = \sum_{m_k \in M_n} p_{kj}^{(2)} p_k^{(2)} + p_{jj}^{(2)} p_j^{(2)} + \sum_{m_i \in M_{n-1},\, i \neq j} p_{ij}^{(2)} p_i^{(2)} \tag{4}$$

subtracting Eqs. (2) and (3), Theorem 3, together with the fact that $p_j^{(3)} = p_j^{(2)} + \Delta p_j$ and noticing that the number of states in M_{n-2} that return to state m_j is $n-1$, we get that for any $m_l \in M_{n-2}$,

$$\Delta p_j = \sum_{m_k \in M_n} p_{kj}^{(2)} \Delta p_k + p_{jj}^{(2)} \Delta p_j - \Theta(hf) p_j^{(2)} - \Theta(hf) \Delta p_j + \sum_{m_i \in M_{n-1}} p_{ij}^{(2)} \Delta p_i + \Theta(n/m) p_l^{(3)}.$$

Selecting $\Delta p_j = \Theta(n/m)p_l^{(3)} = \Theta(hf)p_j^{(2)}$, the equation is satisfied providing that $f = O(1/n^3)$.

In addition, using the above equations, there exists a constant $c_3 < 1$ such that $p_l^{(3)} \leqslant c_3 p_j^{(2)}$.

By repeating the process for states in M_k, $k < n-2$, and using an inductive argument, we get that for every $m \in M_k$,

$$\Pr\{m\} \leqslant c_3^{n-k}\left(\frac{1}{|M_n| + |M_{n-1}|} + \frac{O(fn^2)}{|M_n| + |M_{n-1}|}\right)$$

but due to the uniqueness of the steady-state probabilities, the above inequalities are the only way to satisfy the balance equations, and the theorem is proved. □

It is worth to remark that $\mathcal{M}_3$ is driven by $\mathcal{M}_2$ because it can arrive to states in M_k, $k < n-1$ only by f-transitions and $f = O(1/n^3)$. If instead, we had considered a $\mathcal{M}_4$ in place of $\mathcal{M}_3$ where this new chain in addition to the f-transitions would behave as $\mathcal{M}_2$ behaves with respect to balancing (i.e. when we get a new edge that exists in the matching we delete it) by using a similar argument as the previous one obtain

Theorem 8. *For the steady-state probability of* $\mathcal{M}_4$,

$$\forall m \in \mathcal{M}_4, \quad \Pr m = \frac{\Theta(1)}{|M_n| + |M_{n-1}| + \cdots + |M_1|}.$$

Let us turn into the mixing properties of $\mathcal{M}_3$. We have that $\mathcal{M}_3$ is a non-reversible, strongly aperiodic and ergodic chain. Moreover, as we have shown, the steady-state probability for states in $M_n \cup M_{n-1}$ follows closely the uniform distribution for $M_n \cup M_{n-1}$. Notice that the probabilities for any state in M_k, $k \leqslant n-2$ drop quickly. How fast is $\mathcal{M}_3$ mixing? To answer this question is hard, especially for sets of states in M_k, $k \leqslant n-2$. But we only need $\mathcal{M}_3$ to mix rapidly with respect to perfect matchings and near-perfect matchings.

Definition 4. The Markov chain $\mathcal{M}'_3$ is obtained from the chain $\mathcal{M}_3$ by considering the set of all states $\{m \in M_k \mid k \leqslant n-2\}$ as a single state α. Therefore for $\mathcal{M}'_3$ we get $\Omega = M_n \cup M_{n-1} \cup \{\alpha\}$. The transitions of $\mathcal{M}'_3$ are the induced transitions from $\mathcal{M}_3$.

The next lemma is straightforward to prove,

Lemma 1. *With regard to states in* $M_n \cup M_{n-1}$, *the chains* $\mathcal{M}_3$ *and* $\mathcal{M}'_3$ *have the same steady-state probabilities.*

Theorem 9. *The chain* $\mathcal{M}'_3$ *is rapidly mixing.*

Proof. Notice that due to the ergodicity of $\mathcal{M}_3$ we know that $\mathcal{M}'_3$ is also ergodic. We also have to prove that $\mathcal{M}_3$ is strongly aperiodic. From $m_i \in M_n$, $\pi_{ii} \geqslant \frac{1}{2}$ as in $\mathcal{M}_2$.

For $m_i \in M_{n-1}$, $\pi_{ii} \geqslant \frac{1}{2}$ by definition (where the π's denote the steady-state probabilities). Let us compute $\pi_{\alpha\alpha}$,

$$\pi_{\alpha\alpha} = 1 - \sum_{m\in\alpha,\ m'\in\Omega-\alpha} \pi_m p_{mm'} = 1 - \sum_{m\in m_{n-2},\ m'\in M_{n-1}} \pi_m p_{mm'}.$$

But for $m \in m_{n-2}$ and $m' \in M_{n-1}$ we have that $p_{mm'} = 1/2m$, we also know $\sum_{m\in M_{n-2}} \pi_m \leqslant 1$, therefore

$$\pi_{\alpha\alpha} = 1 - (1/2m) \sum_{m\in M_{n-2}} \pi_m \geqslant 1 - 1/2m > 1/2.$$

By Theorem 4, it remains to look into the conductance of $\mathcal{M}_3'$. Note that $\Phi = \min_{\{B\subseteq\Omega\}} \Phi(B)$, where the B's are such that $\sum_{m_i\in B} \pi_i \leqslant \frac{1}{2}$. Moreover,

$$\Phi(B) = \frac{\sum_{m_i\in B} \sum_{m_j\in\Omega-B} \pi_i p_{ij}}{\sum_{m_i\in B} \pi_i}.$$

Notice that by Theorem 4, we use the strong aperiodicity of the non-reversible $\mathcal{M}_3$ to allow us to use *standard conductance* and not *merging conductance* (see [6] for the definitions).

By Theorem 7 we have that for all $m_i \in M_n \cup M_{n-1}$ in $\mathcal{M}_3$

$$\left| \pi_i - \frac{c}{|M_n| + |M_{n-1}|} \right| = \frac{\Theta(hf)}{|M_n| + |M_{n-1}|} = \frac{\Theta(1/n)}{|M_n| + |M_{n-1}|}$$

and because of the lemma we have that for any $m_i \in \mathcal{M}_3'$, with $m_i \in M_n \cup M_{n-1}$,

$$\frac{c}{|M_n| + |M_{n-1}|} - \frac{\Theta(1/n)}{|M_n| + |M_{n-1}|} \leqslant \pi_i \leqslant \frac{c}{|M_n| + |M_{n-1}|} + \frac{\Theta(1/n)}{|M_n| + |M_{n-1}|}$$

so for every B and every m_i, as $p_{ij} \geqslant 1/\text{poly}(n)$ then

$$\Phi(B) \geqslant \frac{c - \Theta(1/n)}{c + \Theta(1/n)} \cdot \frac{|\text{edges out } B|}{|B|} \cdot \frac{1}{\text{poly}(n)}.$$

If $B = \{\alpha\}$ then $|\text{edges out } B|/|B| \geqslant 1$. For any other B with $\sum_{m_i\in B} \pi_i \leqslant \frac{1}{2}$, we have an underlying graph which is exactly as the one for $\mathcal{M}_3$ and with respect to the one of $\mathcal{M}_2$ it has a few edges added; the ones to and from α. Therefore by Theorem 7 we get $|\text{edges out } B|/|B| \geqslant 1/\text{poly}(n)$. Thus, we have $\Phi(B) \geqslant 1/\text{poly }(n)$ and the theorem is proved. □

As a corollary of Theorem 9, and the previous lemma we get the following result.

Corollary 1. *The Markov chain $\mathcal{M}_3$ is rapidly mixing as far as the states in $M_n \cup M_{n-1}$ is concerned. It is also rapidly mixing with respect to $\sum_{m_i\in\alpha} \pi_i$.*

Let Λ be the probability distribution induced by the experiment of selecting a sequence $\vec{e} = e_1, e_2, \ldots, e_{n+1}$ of $n+1$ random edges from the graph, break it at the first edge where the induced subgraph does not form a matching, and discard the remaining edges.

Notice that Λ is a probability distribution over the *ordered* Ω set of all matchings in the graph.

Let us define another Markov chain $\mathscr{M}_4$ with the same state space as $\mathscr{M}_3$ and transitions defined as follows: Select a matching m according to Λ, and a single transition is the result of following in $\mathscr{M}_3$ the path defined by m.

As a consequence of this definition, we have

Theorem 10. *The chain $\mathscr{M}_4$ is rapidly mixing and its steady-state distribution is the same as $\mathscr{M}_3$.*

5. Conclusions

From now on, the same arguments given in [1, 11], can be used to obtain a fully approximation scheme to the total number of perfect matchings, and therefore to approximate the permanent of a dense boolean matrix.

Similar techniques like the ones developed in this paper can be applied to other counting problems that have been approximated using the Markov approach: Counting 2-factors in dense graphs [3], counting the number of Hamiltonian cycle in dense graphs [2], and counting the number of Eulerian orientations in a graph [7].

We believe that in the future, there could be many independent applications for our treatment of non-uniform, non-reversible chains that are "close" to uniform and also are "partial rapid mixing".

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